3.45 (Angle Chasing)

Refath Bari

7/26/18

## 1 Question

Let ABCD be a square, and let E, F be points such that DA = DE = DF = DC, and  $\angle ADE = \angle EDF = \angle FDC$ . Prove that  $\triangle BEF$  is equilateral.



Figure 1: Square, and a bunch of nice points

Remember, try it on your own before you see my solution. If you see any errors in my solution, please let me know at Bari Tutorials.

## 2 What we know

Let ABCD be a square, and let E, F be points such that DA = DE = DF = DC, and  $\angle ADE = \angle EDF = \angle FDC$ . Prove that  $\triangle BEF$  is equilateral.



Figure 2: We've trying to prove  $\triangle BEF$ 

Remember, try it on your own before you see my solution. If you see any errors in my solution, please let me know at Bari Tutorials.

## 3 The Auxiliaries!

What can be better than this? To solve the problem, we need some auxiliary lines AKA, extra figures added to a problem to make it easier to solve. It took me a long time to figure this one out, but once you do, it's worth it. Let's first draw in line FA. Why? Well, the goal is to prove  $\triangle FEA \cong \triangle BEA$ . If we can do so, we can show that BE = EF!



Figure 3: We've trying to prove  $\triangle BEF$ 

But how do we prove  $\triangle FEA \cong \triangle BEA$ ? Well, first things first, we're know that  $\angle ADE = \angle EDF = \angle FDC$ . Thus, we conclude that

$$\angle ADE = \angle EDF = \angle FDC = \frac{90}{3} = 30^{\circ}$$

Noice. Now, let's do some angle-chasing, and see what we can find that can help us prove  $\triangle FEA \cong \triangle BEA$ . Let us consider  $\triangle AFD$ . Realize that it is isoceles, since AD = FD. Thus,

$$\angle DAF = \angle DFA = \frac{180^o - 60^o}{2} = 60^o$$



Figure 4: Point G is the intersection of lines FA and ED

Great. Now, it should be obvious that we have  $\triangle DAG$  and  $\triangle DFG$  as 30-60-90 triangles. Realize, also that  $\triangle DEF$  and  $\triangle DEA$  are both isosceles. Thus, we conclude that:

$$\angle DAE = \angle DEA = \angle DEF = \angle DFE = \frac{180^{O} - 30^{O}}{2} = 75^{O}$$

We can thus conclude that,

$$\angle GFE = \angle GAE = 75^O - 60^O = 15^O$$

Furthermore,

$$\angle BAE = \angle DAB - \angle EAD = 90^{\circ} - 75^{\circ} = 15^{\circ}$$

Noice! So now, we know that  $\angle BAE \cong \angle EAF$ . Great, one step closer to proving  $\triangle FEA \cong \triangle BEA$ . In fact, we also that AE = AE, by the Reflexive property (or by common sense).



Figure 5: Woah.

OK, folks. Things are about to get greasy, so hold the railings. We know the  $\triangle AFD$  is equilateral, so let's use that to our advantage. Call side AF x. Since AF = AD = AB, we now know that side AB = AF! Yes! Let's go! We can thus prove that  $\triangle FEA \cong \triangle BEA$  by SAS (side AF = AB,  $\angle BAE \cong \angle EAF$ , and side AE = AE).

Thus, be CPCTC, sides EF = EB. But WAIT. That only shows that  $\triangle EFB$  is isoceles, not equilateral! Well, that's why we must construct a second auxiliary line, as shown:



Figure 6: Magic. Well, I'll leave you at that. It's up to you to show that  $\triangle EFC \cong \triangle BFC$ . Save the day!